

# $2^n$ -rational maps

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December 3, 2015

## Abstract

We present a natural extension of the notion of *nondegenerate rational maps* (*quadrirational maps*) to arbitrary dimensions. We refer to these maps as  $2^n$ -rational maps. In this note we construct a rich family of  $2^n$ -rational maps. These maps by construction are involutions and highly symmetric in the sense that the maps and their companion maps have the same functional form.

## 1 Introduction

In [1] Etingof introduced the notion of *nondegenerate rational maps*. These maps arose in the interplay between studies on *set-theoretical solutions of the quantum Yang-Baxter equation* [2, 3] and the theory of geometric crystals [4]. Set-theoretical solutions of the quantum Yang-Baxter equation were introduced in [5, 6]. In [7], among various connections with integrability, the name *Yang-Baxter maps* was proposed instead of set-theoretical solutions. Also, instead of the term nondegenerate rational maps, the name *quadrirational maps* was coined in [8]. We find the terminology of quadrirational maps more adequate for this note, hence we will use it from now on. In recent years many results on the connection between quadrirational Yang-Baxter maps and the theory of discrete integrable systems were obtained [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

A rational map  $R : \mathbb{CP}^1 \times \mathbb{CP}^1 \ni (x, y) \mapsto (X, Y) \in \mathbb{CP}^1 \times \mathbb{CP}^1$  is called quadrirational if the maps  $R^{-1} : (X, Y) \mapsto (x, y)$ ,  $R^c : (X, y) \mapsto (x, Y)$ , and  $(R^c)^{-1} : (x, Y) \mapsto (X, y)$  are rational maps as well. Here,  $R^{-1}$  is the inverse of the map  $R$  and  $R^c$  the so-called *companion map* of  $R$ . If all maps and the companion maps have the same functional form, then we refer to it as quadrirationality in the narrow sense. This note is about quadrirationality and moreover on its generalisation to  $n$  dimensions,  $2^n$ -rationality (see Definition 3.1), mainly in the narrow sense.

The QRT map was introduced by Quispel, Roberts and Thompson [22] as a family of integrable maps on the plane. In [23] Tsuda derived the conditions that led to periodic QRT maps. Although in [23] there is no mention of quadrirationality, it turns out that the period 2 QRT map is a quadrirational one in the narrow sense. An alternative way to introduce quadrirationality via factorisation of involutions was presented in [24] (see also [25]).

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In Section 2, we are recalling the QRT construction and we present some quadrirational subclasses of it. In Section 3 we present a QRT generalisation which was introduced in [23]. Next we consider a natural extension of quadrirationality to arbitrary dimensions which we refer to as  $2^n$ -rationality. We arrive to an explicit form of  $2^n$ -rational maps in the narrow sense. Afterwards we present in detail the  $n = 2$  and  $n = 3$  cases i.e. quadrirational and octarational maps respectively. Finally, in Section 4 we end this article with some conclusions and perspectives.

## 2 Quadrirational QRT Maps

The QRT map [22]  $\phi : \mathbb{CP}^1 \times \mathbb{CP}^1 \ni (x, y) \mapsto (X, Y) \in \mathbb{CP}^1 \times \mathbb{CP}^1$  is defined by the composition of two non-commuting involutions  $\sigma^1, \sigma^2$ , which both preserve the same invariant

$$H(x, y) = \frac{\mathbf{X}^T A_0 \mathbf{Y}}{\mathbf{X}^T A_1 \mathbf{Y}},$$

where  $\mathbf{X}, \mathbf{Y}$  are vectors  $\mathbf{X} = (x^2, x, 1)^T$ ,  $\mathbf{Y} = (y^2, y, 1)^T$  and  $A_0, A_1$  are two  $3 \times 3$  matrices,

$$A_i = \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ \delta_i & \epsilon_i & \zeta_i \\ \kappa_i & \lambda_i & \mu_i \end{pmatrix}.$$

The QRT is the composition  $\sigma^2 \circ \sigma^1$  of the non-commuting involutions  $\sigma^1, \sigma^2$ . The involution  $\sigma^1$  is obtained from the solution of the equation  $H(X, y) - H(x, y) = 0$  for  $X$  and the involution  $\sigma^2$  from the solution of  $H(x, Y) - H(x, y) = 0$  for  $Y$ . Each of these equations has 2 solutions. One solution is the identity map and by choosing the other one we arrive at

$$\sigma^1 : (x, y) \mapsto (X, Y) = \left( \frac{f_1(y) - f_2(y)x}{f_2(y) - f_3(y)x}, y \right), \quad \sigma^2 : (x, y) \mapsto (X, Y) = \left( x, \frac{g_1(x) - g_2(x)y}{g_2(x) - g_3(x)y} \right) \quad (1)$$

where

$$\begin{aligned} (f_1(y), f_2(y), f_3(y))^T &= (A_0 \mathbf{Y}) \times (A_1 \mathbf{Y}), \\ (g_1(x), g_2(x), g_3(x))^T &= (A_0^T \mathbf{X}) \times (A_1^T \mathbf{X}). \end{aligned}$$

To recapitulate, the QRT map is defined by the equations

$$H(x, y) = H(X, y) = H(X, Y). \quad (2)$$

The QRT map preserves a linear pencil of biquadratic curves  $h(x, y; t) = \mathbf{X}^T A_0 \mathbf{Y} - t \mathbf{X}^T A_1 \mathbf{Y} = 0$ . Base points of a linear pencil of biquadratic curves, are the points which are contained in all curves of the linear pencil. They are determined by

$$x = \frac{f_1(y)}{f_2(y)} = \frac{f_2(y)}{f_3(y)}, \quad \text{or} \quad y = \frac{g_1(x)}{g_2(x)} = \frac{g_2(x)}{g_3(x)}. \quad (3)$$

For a given bi-quadratic polynomial, quantities that are covariant under Möbius transformations action on the bi-quadratic under consideration are called relative invariants of the latter. They are given by the

formulas

$$i_2(h, x, y) = 2hh_{xxyy} - 2h_xh_{xyy} - 2h_yh_{yxx} + 2h_{xx}h_{yy} + h_{xy}^2,$$

$$i_3(h, x, y) = \frac{1}{4} \det \begin{pmatrix} h & h_x & h_{xx} \\ h_y & h_{xy} & h_{xxy} \\ h_{yy} & h_{xyy} & h_{xxyy} \end{pmatrix} = \det(A_0 - tA_1).$$

**Proposition 2.1** *For  $i_3 = 0$  the associated QRT involutions  $\sigma^1, \sigma^2$  commute and the QRT map is an involution.*

**Proof 2.1** *There is*

$$i_3 = \det(A_0 - tA_1) = -t^3 \det A_1 + t^2 \left( \frac{\det(A_0 + A_1) + \det(A_0 - A_1)}{2} - \det A_0 \right) \\ + t \left( \frac{\det(A_0 - A_1) - \det(A_0 + A_1)}{2} - \det A_1 \right) + \det A_0 = 0.$$

*Demanding the coefficients of this polynomial on  $t$  to be zero we obtain:*

$$\det A_0 = \det A_1 = 0 = \det(A_0 - A_1) = \det(A_0 + A_1). \quad (4)$$

*Note that  $i_3$  is by definition covariant under Möbius transformations. By a certain Möbius transformation we can send 2 of the base points of the linear pencil to  $(0, 0)$  and  $(\infty, \infty)$ . Then the parameters matrices*

*have the form:  $A_i = \begin{pmatrix} 0 & \alpha_i & \zeta_i \\ \beta_i & \epsilon_i & \gamma_i \\ \eta_i & \delta_i & 0 \end{pmatrix}$   $i = 0, 1$  and the conditions (4) are exactly the conditions obtained*

*by Tsuda (see Theorems 2.4 and 3.4 in [23]) for the QRT to be periodic of period 2, from which it follows that the QRT involutions commute. Hence for  $i_3 = 0$  the associated QRT involutions  $\sigma^1, \sigma^2$  commute and the QRT map is an involution.*

From now on we focus on the following solution of the conditions (4):

$$A_i = \begin{pmatrix} 0 & \alpha_i & 0 \\ \beta_i & \epsilon_i & \gamma_i \\ 0 & \delta_i & 0 \end{pmatrix}, \quad i = 0, 1. \quad (5)$$

The QRT map for this choice of parameter matrices, was given explicitly in [23]. In the following Proposition we repeat a result of [23] and we prove our observation i.e. in this case the period 2 QRT map is a quadrirational map.

**Proposition 2.2** *The QRT map associated to the integral*

$$H(x, y) = \frac{\alpha_0 + \alpha_1 x + \alpha_2 y + \frac{\beta_1}{x} + \frac{\beta_2}{y}}{\gamma_0 + \gamma_1 x + \gamma_2 y + \frac{\delta_1}{x} + \frac{\delta_2}{y}} \quad (6)$$

*reads*

$$\phi : (x, y) \mapsto (X, Y) = \left( \frac{1}{x} \frac{\beta_1 - \delta_1 H(x, y)}{\alpha_1 - \gamma_1 H(x, y)}, \frac{1}{y} \frac{\beta_2 - \delta_2 H(x, y)}{\alpha_2 - \gamma_2 H(x, y)} \right). \quad (7)$$

*It is an involution and moreover a quadrirational map in the narrow sense i.e. the map the companion map and their inverses have the same functional form.*

**Proof 2.2** It is easy to show that the parameter matrices  $A_i$  associated to the integral (6) are of the form (5), so  $i_3 = 0$  and the QRT map is an involution. For the integral (6), the QRT map is  $\phi : (x, y) \mapsto (X, Y)$  where:

$$\begin{aligned} X &= \frac{(\beta_1\delta_0 - \beta_0\delta_1)y^2 + (\epsilon_1\delta_0 - \epsilon_0\delta_1)y + \gamma_1\delta_0 - \gamma_0\delta_1 - (\alpha_0\delta_1 - \alpha_1\delta_0)yx}{(\alpha_0\delta_1 - \alpha_1\delta_0)y - ((\alpha_1\beta_0 - \alpha_0\beta_1)y^2 + (\alpha_1\epsilon_0 - \alpha_0\epsilon_1)y + \alpha_1\gamma_0 - \alpha_0\gamma_1)x} \\ Y &= \frac{(\alpha_1\gamma_0 - \alpha_0\gamma_1)x^2 + (\epsilon_1\gamma_0 - \epsilon_0\gamma_1)x + \gamma_0\delta_1 - \gamma_1\delta_0 - (\beta_0\gamma_1 - \beta_1\gamma_0)xy}{(\beta_0\gamma_1 - \beta_1\gamma_0)x - ((\alpha_0\beta_1 - \alpha_1\beta_0)x^2 + (\beta_1\epsilon_0 - \beta_0\epsilon_1)x + \beta_1\delta_0 - \beta_0\delta_1)y}. \end{aligned} \quad (8)$$

In terms of the integral  $H$ , mapping (8) simplifies to

$$\phi : (x, y) \mapsto (X, Y) = \left( \frac{1}{x} \frac{\beta_1 - \delta_1 H(x, y)}{\alpha_1 - \gamma_1 H(x, y)}, \frac{1}{y} \frac{\beta_2 - \delta_2 H(x, y)}{\alpha_2 - \gamma_2 H(x, y)} \right). \quad (9)$$

The companion map  $\phi^c$  of the mapping  $\phi$ , is the map  $\phi^c : (X, y) \mapsto (x, Y)$  i.e. from (9) we need to solve for  $x, Y$  in terms of  $X, y$ . From the first equality of (9) we obtain  $xX = \frac{\delta_0 - \delta_1 H(x, y)}{\alpha_0 - \alpha_1 H(x, y)}$ , from (2) we have  $H(x, y) = H(X, y)$ , so  $x = \frac{1}{X} \frac{\delta_0 - \delta_1 H(X, y)}{\alpha_0 - \alpha_1 H(X, y)}$ , and from the second equality of (9)  $Y = \frac{1}{y} \frac{\gamma_0 - \gamma_1 H(X, y)}{\beta_0 - \beta_1 H(X, y)}$ . So the companion map is

$$\phi^c : (X, y) \mapsto (x, Y) = \left( \frac{1}{X} \frac{\delta_0 - \delta_1 H(X, y)}{\alpha_0 - \alpha_1 H(X, y)}, \frac{1}{y} \frac{\gamma_0 - \gamma_1 H(X, y)}{\beta_0 - \beta_1 H(X, y)} \right).$$

The map  $\phi^c$  has the same functional form as  $\phi$  so it is also an involution. Therefore all the 4 maps  $\phi, \phi^{-1}, \phi^c, (\phi^c)^{-1}$  have the same functional form and the map  $\phi$  is quadrirational in the narrow sense.

### 3 $2^n$ -rational maps

Generalisations of the QRT map have been proposed by various authors [26, 27, 23, 28, 29, 30]. In order to derive  $2^n$ -rational maps we focus on the generalisation given in [23] which includes the generalisation given in [26]. Namely, a map  $\phi : (\mathbb{CP}^1)^n \ni (x^1, x^2, \dots, x^n) \mapsto (X^1, X^2, \dots, X^n) \in (\mathbb{CP}^1)^n$  that is defined by the composition of  $n$  non-commuting involutions  $\sigma^1, \sigma^2, \dots, \sigma^n$ , i.e. which preserve the same integral

$$H(x^1, x^2, \dots, x^n) = \frac{\sum_{j_1, j_2, \dots, j_n=0}^2 \alpha_{j_1, j_2, \dots, j_n} (x^1)^{2-j_1} (x^2)^{2-j_2} \dots (x^n)^{2-j_n}}{\sum_{j_1, j_2, \dots, j_n=0}^2 \beta_{j_1, j_2, \dots, j_n} (x^1)^{2-j_1} (x^2)^{2-j_2} \dots (x^n)^{2-j_n}}. \quad (10)$$

Note that with  $(\mathbb{CP}^1)^n$  we denote  $(\mathbb{CP}^1)^n := \underbrace{\mathbb{CP}^1 \times \mathbb{CP}^1 \times \dots \times \mathbb{CP}^1}_{n \text{ times}}$ .

The involution  $\sigma^k$  is defined by the non-trivial solution of  $H(x^1, x^2, \dots, X^k, \dots, x^n) = H(x^1, x^2, \dots, x^n)$ , for  $X^k$ . Similarly for the other  $n-1$  involutions. From now on we will call mapping  $\phi$  as *Tsuda* map. Note that for  $n=2$  we have exactly the QRT map. To recapitulate, the Tsuda map is defined by the equations:

$$\forall m \in \{1 \dots n\}, \quad H(x^1, \dots, x^{m-1}, x^m, x^{m+1}, \dots, x^n) = H(X^1, \dots, X^{m-1}, X^m, x^{m+1}, \dots, x^n) \quad (11)$$

and preserves the linear pencil of manifolds

$$\begin{aligned} h(x^1, x^2, \dots, x^n; t) &= \sum_{j_1, j_2, \dots, j_n=0}^2 \alpha_{j_1, j_2, \dots, j_n} (x^1)^{2-j_1} (x^2)^{2-j_2} \dots (x^n)^{2-j_n} - \\ t \sum_{j_1, j_2, \dots, j_n=0}^2 \beta_{j_1, j_2, \dots, j_n} (x^1)^{2-j_1} (x^2)^{2-j_2} \dots (x^n)^{2-j_n} &= 0. \end{aligned} \quad (12)$$

For  $n = 2$  the pencil of manifolds is a linear pencil of elliptic curves. For  $n = 3$  we have a linear pencil of K-3 surfaces and for  $n \geq 4$  we have Calabi-Yau manifolds.

Before we present the definition of a  $2^n$ -rational map, we introduce some notation. Let  $M$  the set of  $n$  indices,  $M := \{1, \dots, n\}$  and  $m$  any subset of the set  $M$  i.e.  $m := \{m_1, m_2, \dots, m_k\}$ ,  $k \leq n$ . Also  $m^c$  the complement of the set  $m$  so  $m^c := M \setminus m$ . Finally, with  $\mathcal{X}^m$  and  $\mathcal{X}^{m^c}$  we denote the coordinates  $\mathcal{X}^m := (x^1, x^2, \dots, X^{m_1}, x^{m_1+1}, \dots, X^{m_2}, \dots, X^{m_k}, \dots, x^n)$  and  $\mathcal{X}^{m^c} := (X^1, X^2, \dots, x^{m_1}, X^{m_1+1}, \dots, x^{m_2}, \dots, x^{m_k}, \dots, X^n)$ . Now we are ready to give the following definition.

**Definition 3.1** A rational map  $R : (\mathbb{CP}^1)^n \ni (x^1, x^2, \dots, x^n) \mapsto (X^1, X^2, \dots, X^n) \in (\mathbb{CP}^1)^n$  is a  $2^n$ -rational map if all  $2^n$  maps  $R^{c_m} : (\mathbb{CP}^1)^n \ni \mathcal{X}^m \mapsto \mathcal{X}^{m^c} \in (\mathbb{CP}^1)^n$ , are rational maps. The maps  $R^{c_m}$  are called companion maps of  $R$ .

**Proposition 3.1** The Tsuda map  $\phi$  associated to the integral

$$H(x^1, \dots, x^n) = \frac{\alpha_0 + \sum_{i=1}^n \alpha_i x^i + \sum_{i=1}^n \beta_i / x^i}{\gamma_0 + \sum_{i=1}^n \gamma_i x^i + \sum_{i=1}^n \delta_i / x^i}, \quad (13)$$

reads

$$\phi : (x^1, \dots, x^n) \mapsto (X^1, \dots, X^n), \text{ where } X^i = \frac{1}{x^i} \frac{\beta_i - \delta_i H(x^1, \dots, x^n)}{\alpha_i - \gamma_i H(x^1, \dots, x^n)}, \quad i = 1, \dots, n. \quad (14)$$

It is an involution and moreover is a  $2^n$ -rational map in the narrow sense i.e. the map the companion maps and their inverses have the same functional form.

**Proof 3.1** By varying the integral (13), apart the identity solution, the equations

$$\forall k \in \{1, \dots, n\} \quad H(x^1, \dots, X^k, \dots, x^n) - H(x^1, \dots, x^n) = 0$$

has as solutions the involutions  $\sigma^k : (x^1, \dots, x^k, \dots, x^n) \mapsto (x^1, \dots, X^k, \dots, x^n)$ ,  $k \in \{1, \dots, n\}$  where

$$X^k = \frac{1}{x^k} \frac{\beta_k - \delta_k H(x^1, \dots, x^n)}{\alpha_k - \gamma_k H(x^1, \dots, x^n)}, \quad k \in \{1, \dots, n\}. \quad (15)$$

For any 2 involutions  $\sigma^i, \sigma^j$ ,  $i, j \in \{1, \dots, n\}$  of (15), since  $H$  is preserved by both of them, it is easy to show by direct computation that  $(\sigma^i \circ \sigma^j)^2 = \text{id}$ ,  $i, j \in \{1, \dots, n\}$ . Hence for the map  $\phi = \sigma^1 \circ \sigma^2 \circ \dots \circ \sigma^n$ , there is  $\phi^2 = \text{id}$  so  $\phi$  it is an involution and reads:

$$\phi : (x^1, \dots, x^n) \mapsto (X^1, \dots, X^n), \text{ where } X^i = \frac{1}{x^i} \frac{\beta_i - \delta_i H(x^1, \dots, x^n)}{\alpha_i - \gamma_i H(x^1, \dots, x^n)}, \quad i = 1, \dots, n.$$

Moreover  $\phi$  is a  $2^n$ -rational map. To prove the  $2^n$ -rationality, we have to find the companion maps of (14). For the coordinates  $\mathcal{X}^m = (x^1, x^2, \dots, X^{m_1}, x^{m_1+1}, \dots, X^{m_2}, \dots, X^{m_k}, \dots, x^n)$  it holds

$$H(x^1, \dots, x^n) = H(x^1, x^2, \dots, X^{m_1}, x^{m_1+1}, \dots, X^{m_2}, \dots, X^{m_k}, \dots, x^n), \quad (16)$$

since  $H$  is an integral. Solving  $k$  of the equalities of (14) for  $x^i, i \in m$  and using (16) we obtain:

$$x^i = \frac{1}{X^i} \frac{\beta_i - \delta_i H(x^1, x^2, \dots, X^{m_1}, x^{m_1+1}, \dots, X^{m_2}, \dots, X^{m_k}, \dots, x^n)}{\alpha_i - \gamma_i H(x^1, x^2, \dots, X^{m_1}, x^{m_1+1}, \dots, X^{m_2}, \dots, X^{m_k}, \dots, x^n)}, \quad i \in m. \quad (17)$$

Where  $m$  the set of  $k$  indices (see the notation introduced previously). For the remaining  $n - k$  equalities of (14), we have

$$X^j = \frac{1}{x^j} \frac{\beta_j - \delta_j H(x^1, x^2, \dots, X^{m_1}, x^{m_1+1}, \dots, X^{m_2}, \dots, X^{m_k}, \dots, x^n)}{\alpha_j - \gamma_j H(x^1, x^2, \dots, X^{m_1}, x^{m_1+1}, \dots, X^{m_2}, \dots, X^{m_k}, \dots, x^n)}, \quad j \in m^c. \quad (18)$$

So we have the companion maps  $\phi^{c_m} : \mathcal{X}^m \mapsto \mathcal{X}^{m^c}$ , or

$$\phi^{c_m} : (x^1, x^2, \dots, X^{m_1}, x^{m_1+1}, \dots, X^{m_2}, \dots, X^{m_k}, \dots, x^n) \mapsto (X^1, X^2, \dots, x^{m_1}, X^{m_1+1}, \dots, x^{m_2}, \dots, x^{m_k}, \dots, X^n),$$

where the variables  $x^i, i \in m$  are given explicitly in (17), and the variables  $X^j, j \in m^c$  are given in (18).

Clearly there are  $2^n$  such maps  $\phi^{c_m}$ . All these maps have the same functional form as the original map  $\phi$ , hence they are involutions. So mapping  $\phi$  is a  $2^n$ -rational map in the narrow sense.

### 3.1 $n=1$ . The birational case

For  $n = 1$  and if we rename  $x^1$  as  $x$  we have

$$H(x) = \frac{\alpha_0 + \alpha_1 x + \beta_1/x}{\gamma_0 + \gamma_1 x + \delta_1/x} \quad (19)$$

and the map  $\phi$  reads:

$$\phi : x \mapsto X = \frac{1}{x} \frac{\beta_1 - \delta_1 H(x)}{\alpha_1 - \gamma_1 H(x)} = -\frac{\beta_1 \gamma_0 - \alpha_0 \delta_1 + (\beta_1 \gamma_1 - \alpha_1 \delta_1)x}{\beta_1 \gamma_1 - \alpha_1 \delta_1 + (\alpha_0 \gamma_1 - \alpha_1 \gamma_0)x}. \quad (20)$$

Map  $\phi$  is a fraction linear involution and as well a  $2^1$ -rational map (birational). Note that in this case involutivity implies birationality and vice versa. Consider the following matrix  $\tau = \begin{pmatrix} \alpha_0 & \alpha_1 & \beta_1 \\ \gamma_0 & \gamma_1 & \delta_1 \end{pmatrix}$ . Let  $\tau_{ij}$  the determinants of the matrix generated by the  $i$ th and  $j$ th column of  $\tau$  (the minor determinants of  $\tau$  referred to as Plücker coordinates). Then mapping (20) reads  $x \mapsto X = \frac{\tau_{13} - \tau_{32}x}{\tau_{32} - \tau_{12}x}$ , the Möbius involution.

### 3.2 $n=2$ . The quadrirational case

For  $n = 2$  if we rename the variables  $(x^1, x^2)$  as  $(x, y)$  we have

$$H(x, y) = \frac{\alpha_0 + \alpha_1 x + \alpha_2 y + \beta_1/x + \beta_2/y}{\gamma_0 + \gamma_1 x + \gamma_2 y + \delta_1/x + \delta_2/y} \quad (21)$$

and the map  $\phi$  reads:

$$\phi : (x, y) \mapsto (X, Y) = \left( \frac{1}{x} \frac{\beta_1 - \delta_1 H(x, y)}{\alpha_1 - \gamma_1 H(x, y)}, \frac{1}{y} \frac{\beta_2 - \delta_2 H(x, y)}{\alpha_2 - \gamma_2 H(x, y)} \right). \quad (22)$$

Mapping  $\phi$  is involutive and moreover a  $2^2$ -rational map (quadrirational). Consider the following matrix  $\tau = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \beta_1 & \beta_2 \\ \gamma_0 & \gamma_1 & \gamma_2 & \delta_1 & \delta_2 \end{pmatrix}$ . As in the previous subsection,  $\tau_{ij}$  denotes the Plücker coordinates. Then mapping (22) reads:

$$\phi : (x, y) \mapsto \left( X = \frac{\tau_{34}y^2 + \tau_{14}y + \tau_{54} - \tau_{42}xy}{\tau_{42}y - (\tau_{23}y^2 + \tau_{21}y + \tau_{25})x}, Y = \frac{\tau_{52}x^2 + \tau_{51}x + \tau_{54} - \tau_{35}xy}{\tau_{35}x - (\tau_{23}x^2 + \tau_{13}x + \tau_{43})y} \right). \quad (23)$$

### 3.3 n=3. The octarational case

For  $n = 3$  if we rename the variables  $(x^1, x^2, x^3)$  as  $(x, y, z)$  we have

$$H(x, y, z) = \frac{\alpha_0 + \alpha_1x + \alpha_2y + \alpha_3z + \beta_1/x + \beta_2/y + \beta_3/z}{\gamma_0 + \gamma_1x + \gamma_2y + \gamma_3z + \delta_1/x + \delta_2/y + \delta_3/z} \quad (24)$$

and the map  $\phi$  reads:

$$\phi : (x, y, z) \mapsto (X, Y, Z) = \left( \frac{1}{x} \frac{\beta_1 - \delta_1 H(x, y, z)}{\alpha_1 - \gamma_1 H(x, y, z)}, \frac{1}{y} \frac{\beta_2 - \delta_2 H(x, y, z)}{\alpha_2 - \gamma_2 H(x, y, z)}, \frac{1}{z} \frac{\beta_3 - \delta_3 H(x, y, z)}{\alpha_3 - \gamma_3 H(x, y, z)} \right). \quad (25)$$

Mapping  $\phi$  is involutive and moreover a  $2^3$ -rational map (octarational). If we consider the matrix  $\tau = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \beta_1 & \beta_2 & \beta_3 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \delta_1 & \delta_2 & \delta_3 \end{pmatrix}$ , mapping (25), in terms of the determinants  $\tau_{ij}$  reads:

$$\phi : (x, y, z) \mapsto (X, Y, Z),$$

where

$$\begin{aligned} X &= \frac{\tau_{35}y^2z + \tau_{45}yz^2 + \tau_{15}yz + \tau_{65}z + \tau_{75}y + \tau_{25}xyz}{\tau_{52}yz + (\tau_{32}y^2z + \tau_{42}yz^2 + \tau_{12}yz + \tau_{72}y + \tau_{62}z)x}, \\ Y &= \frac{\tau_{64}xz^2 + \tau_{62}x^2z + \tau_{61}xz + \tau_{67}x + \tau_{65}z + \tau_{63}xyz}{\tau_{36}xz + (\tau_{34}xz^2 + \tau_{32}x^2z + \tau_{31}xz + \tau_{35}z + \tau_{37}x)y}, \\ Z &= \frac{\tau_{72}x^2y + \tau_{73}xy^2 + \tau_{71}xy + \tau_{75}y + \tau_{76}x + \tau_{74}xyz}{\tau_{47}xy + (\tau_{42}x^2y + \tau_{43}xy^2 + \tau_{41}xyz + \tau_{46}x + \tau_{45}y)z}. \end{aligned} \quad (26)$$

## 4 Conclusions

In this note we present a rich family of  $2^n$ -rational maps. These maps by construction are involutions and highly symmetric in the sense that the maps and their companion maps have the same functional form.

$2^n$ -rational maps, are quite exceptional objects inside the set of rational maps. Although on their own are interesting as mathematical entities we expect to have various connections with discrete integrable systems. It is known that the quadrirational maps are related to Yang-Baxter maps and therefore to 2-dimensional difference integrable systems [8, 12]. A subclass of octarational maps provides solutions to the functional tetrahedron equation [31]. Our next goal is to isolate solutions of functional tetrahedron equations within the family of maps (25). We also anticipate the connection of  $2^n$ -rational maps presented here (14) with solutions of higher *simplex* equations [32, 33, 34].

**Acknowledgements** P. K and P. D. would like to thank the Cyprus Research Promotion Foundation for their support through project number KY-ROY/0713/27.

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